JOURNAL OF APPROXIMATION THEORY 13, 491–494 (1975)

Note

On Saturation with Splines

O. SHISHA

Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881

DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTH-FIFTH BIRTHDAY

1.

There is a whole class of results stating that if one can approximate a function very closely by splines of degree $\leqslant k$, then the function reduces essentially to a polynomial of degree $\leqslant k$. One variant of this class is the theorem below for which we give a proof using an argument employed in [11], [12] and [13].

2.

Let k be an integer $\geqslant 0$, and let $-\infty < a < b < \infty$. For $n = 1, 2, \ldots$ let $x_j^{(n)} = a + j(b - a)\, n^{-1}, j = 0, 1, \ldots, n$, so that $I_{n,j} = (x_{j-1}^{(n)}, x_j^{(n)}), j = 1, 2, \ldots, n$, are n congruent subintervals of (a, b). For $n = 1, 2, \ldots$ let S_n be the set of all real functions defined on $[a, b] - \{x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}\}$ which in each $I_{n,j}$, $j = 1, 2, \ldots, n$, coincide with a polynomial of degree $\leqslant k$.

THEOREM. *Let f be a real function defined on (a, b). Suppose, for $n = 1, 2, \ldots$, there is an $s_n \in S_n$ such that, as $n \to \infty$,*

$$\sup_{x \in [a,b] - \{x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}\}} |f(x) - s_n(x)| = o(1/n^{k+1}).$$

Then f coincides on (a, b) with a polynomial of degree $\leqslant k$.

3.

Proof. We assume, as we may, $a = 0$, $b = 1$. Let

$$\epsilon_n = n^{k+1} \sup_{x \in [0,1] - \{0, 1/n, 2/n, \ldots, 1\}} |f(x) - s_n(x)|, \qquad n = 1, 2, \ldots,$$

640/13/4-11

so that $\epsilon_n \to 0$. Then, for $n = 1, 2, \ldots$, $m = 1, 2, \ldots$, $m > n$, we have (with $s_n(0) = \lim_{x \to 0+0} s_n(x)$)

$$| s_m(0) - s_n(0)| \leqslant \sup_{0<x<1/m} |\{f(x) - s_n(x)\} - \{f(x) - s_m(x)\}|$$

$$\leqslant (\epsilon_n/n^{k+1}) + (\epsilon_m/m^{k+1}) \to 0,$$

as $n, m \to \infty$. Hence $s_n(0)$ has a finite limit which we denote $f(0)$. Then, for $n = 1, 2, \ldots, | f(0) - s_n(0)| = \lim_{m \to \infty} | s_m(0) - s_n(0)| \leqslant \epsilon_n/n^{k+1}$. We define similarly $s_n(1)$, $n = 1, 2, \ldots$, and $f(1)$ and have, for $n = 2, 3, \ldots$,

$$\sup_{x\in[0,1]-\{1/n,2/n,\ldots,(n-1)/n\}} | f(x) - s_n(x)| = \epsilon_n/n^{k+1}.$$

We prove now that f is continuous in $[0, 1]$. Let $0 \leqslant c \leqslant 1$ and $0 < \epsilon \leqslant 3$. Choose an integer $n > (3/\epsilon)^{1/(k+1)}$ such that

$$\sup_{x\in[0,1]-\{1/n,2/n,\ldots,(n-1)/n\}} | f(x) - s_n(x)| \leqslant 1/n^{k+1}$$

and such that c is not of the form j/n, $0 < j < n$, j an integer. Let j_1 be the integer satisfying $(j_1 - 1)/n \leqslant c \leqslant j_1/n$, $1 \leqslant j_1 \leqslant n$. There is a $\delta > 0$ such that if $| h | < \delta$ and $c + h \in ((j_1 - 1)/n, j_1/n)$, then $| s_n(c + h) - s_n(c)| < \epsilon/3$. For such an h,

$$| f(c + h) - f(c)| \leqslant | f(c + h) - s_n(c + h)| + | s_n(c + h) - s_n(c)|$$
$$+ | s_n(c) - f(c)| < (2/n^{k+1}) + (\epsilon/3) < \epsilon.$$

Let $-\infty < \alpha < \beta < \infty$ and let g be a real function defined and bounded on $[\alpha, \beta]$. For x and $x + (k + 1)h$ in $[\alpha, \beta]$ consider the $(k + 1)$th difference

$$\Delta_h^{k+1}g(x) = \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} g(x + jh).$$

It is 0 if g coincides in $[\alpha, \beta]$ with a polynomial of degree $\leqslant k$. For every $t \geqslant 0$ set

$$\omega_{k+1}(g; \alpha, \beta; t) = \sup_{\substack{\alpha\leqslant x\leqslant\beta \\ \alpha\leqslant x+(k+1)h\leqslant\beta \\ |h|\leqslant t}} | \Delta_h^{k+1}g(x)|.$$

A well-known tool we shall use is the fact [14, p. 104] that

$$\lim_{t\to 0+0} \omega_{k+1}(g; \alpha, \beta; t)/t^{k+1} = 0$$

implies that $\omega_{k+1}(g; \alpha, \beta; t) = 0$ for every $t \geqslant 0$.

Let $0 < |h| \leqslant t$, $0 \leqslant x \leqslant 1$, $0 \leqslant x + (k+1)h \leqslant 1$. Following [11], [12] and [13], let n_0 be the largest positive integer n for which the closed line-segment L joing x to $x + (k+1)h$ lies in some interval $[(j-1)/n, j/n]$, $1 \leqslant j \leqslant n$. Then $(k+1)|h| > (6n_0)^{-1}$. For otherwise if say $L \subseteq I_0 = [(j_0 - 1)/n_0, j_0/n_0]$, $1 \leqslant j_0 \leqslant n_0$, then L would lie either in one of the two closed halves of I_0 or in its (open) middle third. In each case the maximality of n_0 is contradicted.

Extend the definition of s_{n_0} (if $n_0 > 1$) so as to be continuous in I_0. Since s_{n_0} coincides there with a polynomial of degree $\leqslant k$, we have $\Delta_h^{k+1} s_{n_0}(x) = 0$ and hence

$$|\Delta_h^{k+1} f(x)| = |\Delta_h^{k+1}[f(x) - s_{n_0}(x)]| \leqslant \sum_{j=0}^{k+1} \binom{k+1}{j} |f(x+jh) - s_{n_0}(x+jh)|$$

$$\leqslant 2^{k+1} \sup_{(j_0-1)/n_0 < y < j_0/n_0} |f(y) - s_{n_0}(y)| \leqslant 2^{k+1} \epsilon_{n_0} / n_0^{k+1}$$

$$\leqslant \epsilon_{n_0} [12(k+1)|h|]^{k+1} \leqslant \eta(t)\, t^{k+1},$$

where

$$\eta(t) = [12(k+1)]^{k+1} \max_{n > [6(k+1)t]^{-1}} \epsilon_n \to 0, \qquad \text{as} \qquad t \to 0 + 0.$$

Thus for every $t > 0$,

$$\omega_{k+1}(f; 0, 1; t)/t^{k+1} \leqslant \eta(t)$$

and therefore $\omega_{k+1}(f; 0, 1; t) = 0$ for every $t \geqslant 0$. Hence $\Delta_h^{k+1} f(x) = 0$ whenever $0 \leqslant x \leqslant 1$, $0 \leqslant x + (k+1)h \leqslant 1$ and therefore

$$\Delta_h^j f(x) = 0 \quad \text{whenever} \quad j \geqslant k+1, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant x + jh \leqslant 1. \tag{1}$$

This together with the continuity of f in [0, 1] imply that f coincides there with a polynomial of degree $\leqslant k$. (For [7, Chapter I] the sequence of Bernstein polynomials

$$B_n(x) \equiv \sum_{j=0}^{n} f\left(\frac{j}{n}\right) \binom{n}{j} x^j (1-x)^{n-j}$$

$$\equiv \sum_{j=0}^{n} [\Delta_{1/n}^j f(0)] \binom{n}{j} x^j, \qquad n = 1, 2, \ldots,$$

converges uniformly to $f(x)$ in [0, 1]. By (1),

$$B_n(x) \equiv \sum_{j=0}^{k} [\Delta_{1/n}^j f(0)] \binom{n}{j} x^j, \qquad n = k+1, k+2, \ldots,$$

and hence $f(x) \equiv \lim_{n \to \infty} B_n(x)$ must coincide in [0, 1] with a polynomial of degree $\leqslant k$.)

References

1. J. H. Ahlberg, E. N. Nilson and J. L. Walsh, Complex cubic splines, *Trans. Amer. Math. Soc.* **129** (1967), 391–413.
2. J. H. Ahlberg, E. N. Nilson and J. L. Walsh, "The Theory of Splines and Their Applications," Academic Press, New York and London, 1967.
3. G. Birkhoff, Local spline approximation by moments, *J. Math. Mech.* **16** (1967), 987–990.
4. R. DeVore and F. Richards, Saturation and inverse theorems for spline approximation, *in* "Spline Functions and Approximation Theory," pp. 73–82, Birkhäuser Verlag, Basel and Stuttgart, 1973.
5. R. DeVore and F. Richards, The degree of approximation by Chebyshevian splines, *Trans. Amer. Math. Soc.* **181** (1973), 401–418.
6. D. Gaier, Saturation bei Spline-Approximation und Quadratur, *Numer. Math.* **16** (1970), 129–140.
7. G. G. Lorentz, "Bernstein Polynomials," University of Toronto Press, Toronto, 1953.
8. F. Richards, On the saturation class for spline functions, *Proc. Amer. Math. Soc.* **33** (1972), 471–476.
9. K. Scherer, On the best approximation of continuous functions by splines, *SIAM J. Numer. Anal.* **7** (1970), 418–423.
10. K. Scherer, Characterization of generalized Lipschitz classes by best approximation with splines, *SIAM J. Numer. Anal.* **11** (1974), 283–304.
11. O. Shisha, A characterization of functions having Zygmund's property, *J. Approximation Theory* **9** (1973), 395–397.
12. O. Shisha, Characterization of smoothness properties of functions by means of their degree of approximation by splines, *J. Approxmiation Theory* **12** (1974), 365–371.
13. O. Shisha, On the degree of approximation by step functions, *J. Approximation Theory* **12** (1974), 435–436.
14. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," Macmillan, New York, 1963.